

Pseudo-Retract Functors for Local Lattices and Bifinite L-Domains

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Abstract

Recently, a new category of domains used for the mathematical foundations of denotational semantics, that of L-domains, has been under study. In this paper we consider a related category of posets, that of *local lattices*. First, a completion operator taking posets to local lattices is developed, and then this operator is extended to a functor from posets with embedding-projection pairs to local lattices with embedding-projection pairs. The result of applying this functor to a local lattice yields a local lattice isomorphic to the first; this functor is a pseudo-retract.

Using the functor into local lattices, a continuous pseudo-retraction functor from ω -bifinite posets to ω -bifinite L-domains can be constructed. Such a functor takes a universal domain for the ω -bifinite posets to a universal domain for the ω -bifinite L-domains. Moreover, the existence of such a functor implies that, from the existence of a saturated universal domain for the ω -algebraic bifinites, we can conclude the existence of a saturated universal domain for the ω -bifinite L-domains.

1 Introduction

In the search for structures for use as mathematical foundations of the denotational semantics of programming languages, attention has for several years been focused on the category of bounded-complete (ω -algebraic) directed-complete posets. In particular, the requirement was made that every bounded subset of the poset should have a least upper bound. If one thinks of the points in the poset as being partial information, and of one point as being greater than another if it contains more information, then we are insisting that every demonstrably consistent set of information should have a join in the whole poset of information.

However, recently a new class of posets has been studied as an alternative, namely the (ω -algebraic) L-domains. An L-domain is a directed-complete poset (a cpo) with the property that the set of elements below each element forms a complete lattice. So how do L-domains differ from bounded-complete cpo's? In an L-domain, bounded sets

need not have joins in the poset as a whole. We only require that such a bounded set have a unique minimal upper bound below any given upper bound. Thinking in terms of information, we require a demonstrably consistent collection of information have a join relative to a given witness of this consistency, relative to each particular way we have more complete information. The idea of L-domains was introduced by Achim Jung in his investigations into extensions of Smyth's Theorem [Jun88b, Jun88a]. They were independently discovered by Thierry Coquand as a special instance of his categories of embeddings [Coq88].

For the first part of this paper, we will focus our attention on a class of posets more general than the L-domains. The class we shall focus attention on is that of *local lattices*. A local lattice is a poset with the requirement that each principal ideal (the set of points below a given one) is a complete lattice. From the point of view of computation this is an unusual class to be looking at because we are not making the requirement that our posets be directed-complete. However, in the end we will restrict to finite posets (and then go to the ω -bifinites), and finite local lattices are the same thing as finite L-domains.

2 From Posets to Local Lattices

A classic example of a completion operator on posets is given by the following:

Definition 2.1 Let P be a poset. Given $X \subseteq P$, let

$$\underline{X} = \{y \mid y \leq x \text{ for all } x \in X\} = \{y \mid y \leq X\},$$

i.e., the set of all lower bounds for X in P . Let

$$M(P) = \{\underline{X} \mid X \subseteq P\}$$

ordered by subset inclusion (*i.e.* $\underline{X} \leq \underline{Y}$ iff $\underline{X} \subseteq \underline{Y}$). Let $j : P \rightarrow M(P)$ by $j(x) = \underline{\{x\}} = \downarrow x$ the principal ideal generated by x . ■

This construction is known as the MacNeille completion. Some properties which this construction has are:

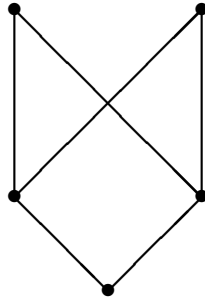
1. $M(P)$ is a complete lattice.
2. $M(P) \cong P$ iff P is a complete lattice.
3. j is an injective function which preserves and reflects order, and which preserves all existing meets and joins.

(See Theorem III.3.11 in [Joh82], for example.) In this section, we shall develop a similar construction for local lattices.

Definition 2.2 A poset L is called an *local lattice* if for every $x \in L$ the principal ideal generated by x (given by $\downarrow x = \underline{x}$) forms a complete lattice under the restricted ordering. ■

Thus we have that an L-domain is a cpo which is also a local lattice. A fairly typical example of a local lattice is the following:

Example 2.3



Notice that, while it is true that any bounded subset of a local lattice must have a meet, as the previous example shows, it need not be the case that a bounded subset have a join (*i.e.* a local lattice need not be bounded complete). However, what will be the case is that below any upper bound for a given bounded subset there will exist a unique minimal upper bound. That is, while we do not have absolute joins for bounded subsets, we do have “joins” for bounded subsets relative to given upper bounds.

Definition 2.4 Let P be a poset, X a bounded subset of P , and u an upper bound for X . Then v is a *join for X relative to u* if v is the join of X in the poset $\downarrow u$ with the restricted order relation. ■

Note that an element v is the join of a bounded set X iff v is the join of X relative to every upper bound for X .

The following gives us a useful way of thinking about local lattices.

Lemma 2.5 For any poset P the following are equivalent:

1. Every non-empty bounded subset has a meet.
2. For each bounded subset and each upper bound of it, there exists a join for the bounded subset relative to the upper bound.
3. P is a local lattice. ■

A complete lattice is a poset in which every subset has a meet. One way of thinking about the MacNeille completion is that it parsimoniously adds meets for subsets of the original poset. (For any subset $X \subseteq P$, the new set \underline{X} will be the meet of $j^*(X) = \{j(x) \mid x \in X\}$ in $M(P)$.) Given the above lemma, a first attempt at turning a poset into a local lattice might be to add, in a similar fashion, meets for bounded subsets. This construction does not quite work. One of the difficulties is that, in adding meets for all bounded subsets of the original poset, you may create

new bounded subsets without meets. While some sense can be made out of iterating the process (by injecting each partial result into the MacNeille completion of the original poset), there are other difficulties which arise. (Ultimately, we are aiming for a functor from posets to local lattices and this process just does not mesh well with functions.) A more fruitful approach lies in attempting to add relative joins for bounded subsets in a reasonably parsimonious fashion, and this is the approach we shall pursue.

But how frugal is frugal? If we only include one minimal upper bound for each bounded subset of our original poset, then we will turn it into a bounded-complete poset, and not just a local lattice. On the other hand, if we try to add one minimal upper bound for each existing upper bound of a given bounded subset, we will obviously just get a big mess. The solution is to amalgamate all the upper bounds of a given bounded subset which must end up having the same minimal upper bound in the local lattice we are trying to build, and then for each such amalgamation add one relative join.

Definition 2.6 Let P be a poset. Given a bounded subset X of P , define

$$\overline{X} = \{y \mid y \geq x \text{ for all } x \in X\} = \{y \mid y \geq X\}.$$

That is, \overline{X} is the set of upper bounds for X in P , or the *ceiling* of X in P . For each bounded subset $X \subseteq P$ define an equivalence relation \sim on \overline{X} by the following:

1. If a and b are elements of \overline{X} and $\{a, b\}$ is bounded in P , then $a \sim b$.
2. If $a \sim b$ and $b \sim c$, then $a \sim c$.

That is, we take the transitive closure of the relation of consistency. We say that a and b are *linked in \overline{X}* if $a \sim b$. Let us write $[a]_X$ for the equivalence class of a in \overline{X} under this equivalence relation. ■

If we view a poset as a graph where there is an edge between two points precisely when one is greater than the other, then $[a]_X$ is just the connected component of \overline{X} which contains a . Two elements a and b of \overline{X} are linked iff there exists a chain a_0, \dots, a_n in \overline{X} such that $a = a_0$, $b = a_n$, and for all i , $0 \leq i < n$ either $a_i \leq a_{i+1}$ or $a_i \geq a_{i+1}$. As a consequence, the notion of being linked is preserved by monotone functions.

Lemma 2.7 Let P and Q be posets, and suppose $f : P \rightarrow Q$ is a monotonic function. If X is a bounded subset of P and u and v are two upper bounds which are linked in \overline{X} , then $f(u)$ and $f(v)$ are linked in \overline{Y} for all subsets Y of Q such that $f^*(\overline{X}) \subseteq \overline{Y}$ (where $f^*(\overline{X}) = \{f(w) \mid w \in \overline{X}\}$). ■

As the next lemma shows us, these equivalence classes do unite upper bounds which must have the same relative join.

Lemma 2.8 *Let X be a bounded subset of a local lattice L , and suppose y and z are two upper bounds of X which are linked in \overline{X} . Then a join for X relative to y is a join for X relative to z . ■*

Using equivalence classes of minimal upper bounds, we are now in a position to define our construction.

Definition 2.9 For any poset P , define

$$L(P) = \{A \subseteq P \mid A = [a]_X, X \subseteq P \text{ bounded}, a \in \overline{X}\}.$$

Define a partial ordering \sqsubseteq on $L(P)$ by superset containment. That is, say that $A \sqsubseteq B$ if $A \supseteq B$. ■

The ordering on $L(P)$ is really just the same as the ordering on the upper or Smyth power domain. Note that every element of $L(P)$ is an upwards-closed subset of P . With this ordering on $L(P)$ we are in a position to see that our operator L does as it is supposed to do and turns a poset into a local lattice.

Theorem 2.10 *For any poset P , the poset $L(P)$ is a local lattice. ■*

If $L(P)$ is to be thought of as the completion of P with respect to being a local lattice, we need to be able to view P as living in $L(P)$. That is, we need an injective function $\iota_P : P \rightarrow L(P)$ such that $\iota_P(p) \leq \iota_P(q)$ iff $p \leq q$.

Definition 2.11 For any poset P define $\iota_P : P \rightarrow L(P)$ by $\iota_P(p) = [p]_{\{p\}} = \uparrow p$, the principal filter generated by p . ■

Theorem 2.12 *The function $\iota_P : P \rightarrow L(P)$ is an injective function which preserves and reflects order, (that is, $\iota_P(p) \leq \iota_P(q) \Leftrightarrow p \leq q$), and preserves all existing meets and joins. ■*

If $L(P)$ is to be thought of as a completion of P with respect to being a local lattice, in addition to having ι_P , we need to know that if P is already a local lattice, then L doesn't change it (up to isomorphism).

Theorem 2.13 *For any poset P we have $P \cong L(P)$ iff P is a local lattice. ■*

It might be nice if we had that L were a minimal completion operator. That is, we might like to have that if P is a poset, M is local lattice, and $j : P \rightarrow M$ is an injection preserving and reflecting order and preserving all existing meets and joins, then there exists a monotonic injection $\eta : L(P) \rightarrow M$ such that $j = \eta \circ \iota_P$. (Actually, we would want that η should be an embedding (see the next section)). Unfortunately, this is too much to hope for. However, we do get the following factorization result.

Proposition 2.14 *Given a poset P , a local lattice M , and a monotonic function $f : P \rightarrow M$, there exists a monotonic function $\eta : L(P) \rightarrow M$ such that $f = \eta \circ \iota_P$. Moreover, if $\nu : L(P) \rightarrow M$ is another monotonic function such that $f = \nu \circ \iota_P$, then $\eta(A) \leq \nu(A)$ for all $A \in L(P)$. ■*

3 Dealing with Functions

So far we have built an operator L which takes a poset and turns it into a local lattice, and if the poset is already a local lattice, it leaves it alone (up to isomorphism). Moreover, we have a way of viewing P as living in $L(P)$ via our monotone injective function ι_P . But what we are after is more, namely that L be (extended to) a functor from an appropriate category of posets to a correspondingly appropriate category of local lattices. For both categories the morphisms we will be interested in are embedding-projection pairs.

Definition 3.1 Given posets P and Q , a monotone function $\epsilon : P \rightarrow Q$ is an *embedding* if there exists a monotone function $\rho : Q \rightarrow P$ such that $\rho \circ \epsilon = \text{id}_P$ and $\epsilon \circ \rho(q) \leq q$ for all $q \in Q$. The function ρ is called a *projection* from Q to P . ■

It should be noted that an embedding is an injection and a projection is a surjection. It is a fairly straight-forward and well-known fact (see Proposition 0.3.2 in [GHK*80], for example) that an embedding uniquely determines its corresponding projection, and likewise a projection uniquely determines its corresponding embedding. Therefore, instead of referring to the category of posets with embedding-projection pairs, we could equally well refer to the category of posets with embeddings or to the category of posets with projections.

In order to extend L to a functor from the category of posets with embedding-projection pairs (\mathbf{PO}^{ep}) to the category of local lattices with embedding-projection pairs (\mathbf{LL}^{ep}), we first need to extend L to act on embedding-projection pairs.

For sake of convenience, if $f : X \rightarrow Y$, let $f^* : 2^X \rightarrow 2^Y$ denote the function $f^*(U) = \{f(u) \mid u \in U\}$.

Definition 3.2 Let P and Q be posets, and suppose that $(\epsilon : P \rightarrow Q, \rho : Q \rightarrow P)$ is an embedding-projection pair. Define $(L(\epsilon) : L(P) \rightarrow L(Q), L(\rho) : L(Q) \rightarrow L(P))$ by

$$L(\epsilon)(A) = [\epsilon(a)]_{\epsilon^*(\underline{A})} \quad \text{some } a \in A$$

and

$$L(\rho)(B) = [\rho(b)]_{\rho^*(\underline{B})} \quad \text{some } b \in B. \quad \blacksquare$$

That $L(\epsilon)$ and $L(\rho)$ are well-defined functions, *i.e.*, that their definition is independent of the choices of a and b , is given to us by Lemma 2.7. The next thing we need to know is that if (ϵ, ρ) is an embedding-projection pair, then so is $(L(\epsilon), L(\rho))$.

Lemma 3.3 *If P and Q are posets and $(\epsilon : P \rightarrow Q, \rho : Q \rightarrow P)$ is an embedding-projection pair, then $(L(\epsilon), L(\rho))$ is also an embedding-projection pair. ■*

Just showing that L takes embedding-projection pairs to embedding-projection pairs is not all that we must do. It is important to verify that L also preserves the identity morphisms and that it preserves composition.

Theorem 3.4 Let \mathbf{PO}^{ep} be the category of posets and embeddings-projection pairs (where the arrow points in the direction of the embedding). Let \mathbf{LL}^{ep} be the category of local lattices with embeddings-projection pairs. Then $L : \mathbf{PO}^{ep} \rightarrow \mathbf{LL}^{ep}$ is a functor. ■

We have more from L , namely that it commutes with the action of the ι_P 's.

Lemma 3.5 For every P and $Q \in \mathbf{PO}^{ep}$, and $(\epsilon, \rho) \in \text{Hom}_{\mathbf{PO}^{ep}}(P, Q)$ we have

$$L(\epsilon) \circ \iota_P = \iota_Q \circ \epsilon \quad \text{and} \quad \iota_P \circ L(\rho) = \rho \circ \iota_Q.$$

That is, the following diagrams commute:

$$\begin{array}{ccc} P & \xrightarrow{\epsilon} & Q \\ \downarrow \iota_P & & \downarrow \iota_Q \\ L(P) & \xrightarrow{L(\epsilon)} & L(Q) \end{array} \quad \begin{array}{ccc} Q & \xrightarrow{\rho} & P \\ \downarrow \iota_Q & & \downarrow \iota_P \\ L(Q) & \xrightarrow{L(\rho)} & L(P) \end{array} \quad \blacksquare$$

Definition 3.6 Let \mathbf{C} be a category and let \mathbf{B} be a full subcategory. A functor $F : \mathbf{C} \rightarrow \mathbf{B}$ is a *pseudo-retraction* if there exists a natural isomorphism from the identity functor on \mathbf{B} to F restricted to \mathbf{B} , i.e. if there exists a family of isomorphisms $i_B : B \cong F(B)$, one for each object in \mathbf{B} , such that for all arrows $f : B_1 \rightarrow B_2$ in \mathbf{B} , we have $F(f) \circ i_{B_1} = i_{B_2} \circ f$. ■

Corollary 3.7 The functor $L : \mathbf{PO}^{ep} \rightarrow \mathbf{LL}^{ep}$ is a pseudo-retraction. ■

4 Restricting to the Bifinites

So far we have been looking at the categories of posets and of local lattices. However, for a class of posets to be used as a mathematical foundation for the denotational semantics of programming languages (in order to sensibly model computation), one usually restricts to the class of directed-complete posets (cpo's). Unfortunately, the functor L which we have built does not take cpo's to cpo's; it does not in general preserve directed-completeness. However, it does preserve finiteness.

Recall that an ω -bifinite domain is a directed-complete poset which is an ω -bilimit (directed colimit) of an ω -chain in the category $\mathbf{DCPO}_{\perp}^{ep}$ of directed-complete posets with continuous embedding-projection pairs of finite posets with least element. (An ω -chain in a category \mathbf{C} is a functor $F : \omega \rightarrow \mathbf{C}$ from the ordinal ω viewed as

a category.) In [GJ88] it is observed that the subcategories of ω -bifinite domains ($\omega\mathbf{B}^{ep}$) and ω -bifinite L-domains ($\omega\mathbf{BL}^{ep}$) are closed under the formation of bilimits of ω -chains in the category of directed complete posets with continuous embedding-projection pairs. If we restrict our functor L to the finite posets, we may use it to construct a functor \bar{L} from the ω -bifinite posets to the ω -bifinite L-domains which has much the same behavior as L .

Although we only intend to use the next in this paper in the the context of categories of posets with embedding-projection pairs, it makes sense in a much broader setting, and we shall state it in that broader setting.

Definition 4.1 Let \mathbf{C} be a category with colimits of ω -chains and let \mathbf{B} be a non-empty full subcategory of \mathbf{C} . We shall say that \mathbf{B} is an ω -closed subcategory of \mathbf{C} if \mathbf{B} is closed under the formation of colimits of ω -chains and if every object A of \mathbf{C} is an object of \mathbf{B} whenever there exists an object B of \mathbf{B} and an arrow $f : A \rightarrow B$ in \mathbf{C} . ■

The following lemma is certainly a previously-known fact. However, the author could find no direct reference to it.

Lemma 4.2 Let \mathbf{B} be an ω -closed subcategory of $\omega\mathbf{B}^{ep}$, and let \mathbf{A} be the intersection of \mathbf{B} with the finite posets (i.e. the full subcategory of \mathbf{B} whose objects are exactly the finite posets of \mathbf{B}). Let $I : \mathbf{A} \rightarrow \mathbf{B}$ be the inclusion functor. Suppose that F is a functor from \mathbf{A} to any ω -complete category \mathbf{C} . Then there exists a functor $\hat{F} : \mathbf{B} \rightarrow \mathbf{C}$ which preserves ω -colimits and whose restriction to \mathbf{A} , i.e., $\hat{F} \circ I$, is naturally isomorphic to F . Moreover, any other functor $G : \mathbf{B} \rightarrow \mathbf{C}$ which preserves ω -colimits and whose restriction to \mathbf{A} is naturally isomorphic to F is naturally isomorphic to \hat{F} . ■

Corollary 4.3 Let \mathbf{B} be an ω -closed subcategory of $\omega\mathbf{B}^{ep}$, and let \mathbf{A} be the intersection of \mathbf{B} with the finite posets, $\mathbf{PO}_{<\omega}^{ep}$. Let $I : \mathbf{A} \rightarrow \mathbf{B}$ be the inclusion functor of \mathbf{A} into \mathbf{B} , and let $J : \mathbf{PO}_{<\omega}^{ep} \rightarrow \omega\mathbf{B}^{ep}$ be the inclusion functor of finite posets with embedding-projection pairs into the bifinites. Let $G : \mathbf{PO}_{<\omega}^{ep} \rightarrow \mathbf{A}$ be a pseudo-retraction functor. Then there exists a pseudo-retraction $\bar{G} : \omega\mathbf{B}^{ep} \rightarrow \mathbf{B}$ which preserves ω -bilimits and whose restriction to the finite posets, $\bar{G} \circ J$, is naturally isomorphic to $I \circ G$. Pictorially, we have that the following diagram commutes up to natural isomorphism.

$$\begin{array}{ccc}
 \mathbf{PO}_{<\omega}^{ep} & \xrightarrow{G} & \mathbf{A} \\
 \downarrow J & & \downarrow I \\
 \omega\mathbf{B}^{ep} & \xrightarrow{\bar{G}} & \mathbf{B}
 \end{array}
 \quad \blacksquare$$

In [GJ88], it is also shown that the category of ω -bifinite L-domains has a universal domain.

Definition 4.4 Let \mathbf{C} be a category. An object \mathcal{U} is *universal* in \mathbf{C} if for every object A of \mathbf{C} , there is a (not necessarily unique) arrow $f : A \rightarrow \mathcal{U}$. ■

Notice that it is immediate from the definitions, that if \mathcal{U} is universal in a category \mathbf{C} and $F : \mathbf{C} \rightarrow \mathbf{B}$ is a pseudo-retraction, then $F(\mathcal{U})$ is universal in \mathbf{B} . That a universal domain \mathcal{U} exists for the ω -bifinites was shown by Gunter in [Gun87]. Therefore, we have

Corollary 4.5 Let \mathcal{U} be a universal domain for the ω -bifinites, and let \bar{L} be a continuous extension of the functor L restricted to the finite posets. Then $\bar{L}(\mathcal{U})$ is a universal domain in the category of ω -bifinite L-domains. ■

5 Functors and Universal Domains

As we have already seen, pseudo-retraction functors provide us with an easy means to conclude that a subcategory of ω -bifinites has a universal domain. In [GJ88] the notion of a specific kind of universal domain, that of a *fully-saturated* universal domain, was introduced. As was shown in that paper, fully-saturated universal domains have the advantage that, when they exist, they are unique up to isomorphism. In the remainder of this paper, we shall show that pseudo-retraction functors also provide us with a means to conclude that a subcategory of ω -bifinites has a fully-saturated universal domain.

We begin with some preliminary definitions and facts.

Recall that an object B is said to be finite in a category \mathbf{C} provided that for every functor $F : \mathbf{I} \rightarrow \mathbf{C}$ with colimit (k_I, K) , if there exists an arrow $f : B \rightarrow K$, then there exists an object I of \mathbf{I} and an arrow $h : B \rightarrow F(I)$ such that $f = k_I \circ h$. This is the categorical equivalent of an element being finite (or compact) in a poset. In the various categories in which we are interested, the finite objects are just those of finite cardinality.

Definition 5.1 (Gunter and Jung) An arrow $f : A \rightarrow B$ in a category \mathbf{C} is an *increment* if for every pair of arrows h, g in \mathbf{C} with $f = h \circ g$ either h or g is an isomorphism. A category \mathbf{C} is *incremental* if

1. \mathbf{C} has an initial object,
2. \mathbf{C} has colimits of ω -chains,
3. every object A of \mathbf{C} is a colimit of an ω -chain (A_i, a_{ji}) where A_0 is initial, each A_i is finite (in the category \mathbf{C}), and each arrow $a_{i,i+1} : A_i \rightarrow A_{i+1}$ is an increment. ■

Lemma 5.2 (Gunter and Jung) *Let \mathbf{C} be an incremental category and let \mathbf{B} be an ω -closed full subcategory of \mathbf{C} . Then \mathbf{B} is an incremental category. ■*

In particular, we have that $\omega\mathbf{B}^{ep}$ is incremental, since $\omega\mathbf{B}^{ep}$ is.

Definition 5.3 (Gunter and Jung) Let \mathbf{C} be an incremental category and let A be an object in \mathbf{C} . An object A^+ together with an arrow $s : A \rightarrow A^+$ is a *saturation* of A in \mathbf{C} if, for every increment $f : B \rightarrow B'$ and arrow $g : B \rightarrow A$, there exists an arrow $h : B' \rightarrow A^+$ such that $h \circ f = s \circ g$. That is, there exists an h which makes the following diagram commute.

$$\begin{array}{ccc}
 B & \xrightarrow{f} & B' \\
 g \downarrow & & \downarrow h \\
 A & \xrightarrow{s} & A^+
 \end{array}$$

The category \mathbf{C} is said to have *finite saturations* if for every finite object A in \mathbf{C} there exists a saturation $s : A \rightarrow A^+$ such that A^+ is finite. ■

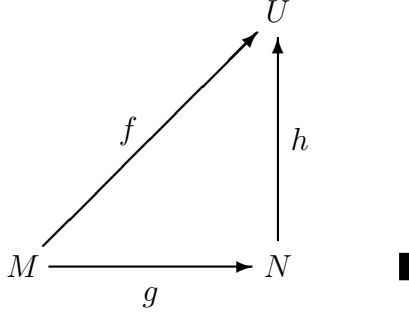
The following fact about saturations follows immediately from the definition.

Lemma 5.4 *Let A, A^+ be objects in an incremental category \mathbf{C} , and let $f : A \rightarrow A^+$ be a saturation of A . Then, for every object C in \mathbf{C} , if there exists an arrow $g : A^+ \rightarrow D$, then $g \circ f : A \rightarrow D$ is another saturation of A and if there exists an arrow $h : C \rightarrow A$, then $f \circ h : C \rightarrow A^+$ is a saturation of C . ■*

In general, the image of a finite saturation under a functor will not be a finite saturation. Even if the functor is a pseudo-retraction, the image of a finite saturation will not generally be a finite saturation. However, as the next lemma shows, a pseudo-retraction will carry some finite saturations to finite saturations.

Lemma 5.5 *Let \mathbf{C} be an incremental category and let \mathbf{B} be an ω -closed full subcategory of \mathbf{C} . Let $F : \mathbf{C} \rightarrow \mathbf{B}$ be a pseudo-retraction of \mathbf{C} to \mathbf{B} . If A is an object in \mathbf{B} and $s : A \rightarrow A^+$ is a saturation of A in \mathbf{C} , then $F(s) : F(A) \rightarrow F(A^+)$ is a saturation of $F(A)$ in \mathbf{B} . ■*

Definition 5.6 (Gunter and Jung) Let \mathbf{B} be an ω -closed full sub-category of $\omega\mathbf{B}^{ep}$. An object U of \mathbf{B} is *fully saturated* in \mathbf{B} if for every pair of objects M, N and arrows $f : M \rightarrow U$ and $g : M \rightarrow N$ in \mathbf{B} there exists an arrow $h : N \rightarrow U$ such that $f = h \circ g$. Thus, we have the following commutative diagram:



From the definition we can see that a fully saturated object is a universal domain. More than that, it was shown in [GJ88] that any fully saturated object is unique up to isomorphism. Also in that paper a means for constructing a fully saturated object was given.

Theorem 5.7 (Gunter and Jung) *Suppose that \mathbf{B} is a closed full sub-category of $\omega\mathbf{B}^{ep}$. Let (S_i, s_{ij}) be an ω -chain in \mathbf{B} where S_0 is initial, each S_i is finite, and each $s_{i+1,i} : S_i \rightarrow S_{i+1}$ is a saturation in \mathbf{B} , and let \mathcal{U} be the bilimit of this ω -chain. Then \mathcal{U} is fully saturated in \mathbf{B} . ■*

Now, a pseudo-retraction functor does not preserve arbitrary finite saturations, so just taking the image of an ω -chain of finite saturations and then taking its bilimit need not get us a fully saturated domain. (In fact, in the case of \bar{L} it will not.) However, a pseudo-retraction does give us a way of constructing an ω -chain of finite saturations out of the image of an ω -chain of finite saturations.

Lemma 5.8 *Let (S_i, s_{ji}) be an ω -chain in $\omega\mathbf{B}^{ep}$ where S_0 is initial, each S_i is finite, and each $s_{i,i+1} : S_i \rightarrow S_{i+1}$ is a saturation. Let \mathbf{B} be an ω -closed full subcategory of $\omega\mathbf{B}^{ep}$, and suppose there exists a pseudo-retraction $F : \omega\mathbf{B}^{ep} \rightarrow \mathbf{B}$ which takes finite objects of $\omega\mathbf{B}^{ep}$ to finite objects of \mathbf{B} . Then there exists a subchain (S_{n_i}, s_{n_j, n_i}) (with $n_i > n_j$ for all $i > j$) such that $F(S_{n_0})$ is initial in \mathbf{B} , and a sequence of embeddings $e_i : F(S_{n_i}) \rightarrow S_{n_{i+1}}$ such that $e_i \circ F(s_{n_{i-1}, n_i}) = s_{n_i, n_{i+1}} \circ e_{i-1}$ and such that $F(e_i) : F^2(S_{n_i}) \rightarrow F(S_{n_{i+1}})$ is a saturation in \mathbf{B} . Pictorially, we have*

$$\begin{array}{ccccc}
S_{n_{i-1}} & \xrightarrow{s_{n_{i-1},n_i}} & S_{n_i} & \xrightarrow{s_{n_i,n_{i+1}}} & S_{n_{i+1}} \\
& \nearrow e_{i-1} & & \nearrow e_i & \\
F(S_{n_{i-1}}) & \xrightarrow{F(s_{n_{i-1},n_i})} & F(S_{n_i}) & \xrightarrow{F(s_{n_i,n_{i+1}})} & F(S_{n_{i+1}}) \\
& \nearrow F(e_{i-1}) & & \nearrow F(e_i) & \\
F^2(S_{n_{i-1}}) & \xrightarrow{F^2(s_{n_{i-1},n_i})} & F^2(S_{n_i}) & \xrightarrow{F^2(s_{n_i,n_{i+1}})} & F^2(S_{n_{i+1}}) \quad \blacksquare
\end{array}$$

Theorem 5.9 *Let \mathbf{B} be an ω -closed full subcategory of $\omega\mathbf{B}^{ep}$, and suppose there exists a pseudo-retraction $F : \omega\mathbf{B}^{ep} \rightarrow \mathbf{B}$ which takes finite objects of $\omega\mathbf{B}^{ep}$ to finite objects of \mathbf{B} . Then there exists a fully saturated object in \mathbf{B} . ■*

Corollary 5.10 *There exists a fully saturated object for the category of ω -bifinite L -domains. ■*

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